UNIVERSITI SAINS MALAYSIA

Final Examination
2015/2016 Academic Session

May/June 2016

JIM 213 – Differential Equations I
[Persamaan Pembezaan I]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **TEN** printed pages before you begin the examination.

Answer **ALL** questions.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEPULUH** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.

1. (a) State the existence and uniqueness theorem for the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y\left(t_0\right) = y_0.$$

Consider the initial value problem

$$(2 - t^2)\frac{dy}{dt} + y = \ln(1 + t), \quad y(0) = -1.$$

Determine the largest interval I such that the solution to this problem is certain to exist.

(40 marks)

(b) Identify the *type* for the following differential equation, hence solve it.

$$\ln x \frac{dy}{dx} + \frac{1}{x} y = \frac{2}{x} \ln x$$

(30 marks)

(c) Find the particular solution to the differential equation

$$e^x \frac{dy}{dx} + xy^2 = 0$$

such that $y \to \frac{1}{2}$ when $x \to \infty$.

(30 marks)

2. (a) Show that

$$\left(2y+2x\right)dy+\left(y^{2}+2xy+2y\right)dx=0$$

is NOT exact differential equation.

By multiplying the equation by e^{x}, show that the equation is now exact. Hence find an implicit solution to the equation

(40 marks)

(b) The motion of the spring mass system is governed by the second order linear homogeneous differential equation

$$y''+\gamma y'+4y=0\,.$$

Determine the values of γ for the system to be critically damped.

For $\gamma=5$, solve the equation with initial condition $y(0)=1,\ y'(0)=2$.

(60 marks)

3. (a) Write down the appropriate form of the particular solution to

$$y''-3y'-4y=3e^{2t}-\sin 4t$$

based on the method of undetermined coefficients. Do **not** attempt to solve the coefficients.

(30 marks)

(b) Solve the Euler differential equation

$$x^{2}\frac{d^{2}y}{dx^{2}}+x\frac{dy}{dx}-y=0$$

Find the general solution of the second order differential equation

$$x^{2}\frac{d^{2}y}{dx^{2}}+x\frac{dy}{dx}-y=x\ln x,$$

by using the method of variation of parameters. State an interval on which the general solution is defined.

(70 marks)

4. Given a system of homogenous linear differential equations

$$\frac{dx}{dt} - x = 0$$
$$\frac{dy}{dt} - y - z = 0$$
$$\frac{dz}{dt} + 2y + z = 0$$

(a) Write the system of equations in the form

$$\frac{dX}{dt} = AX,$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and identify the matrix A.

(15 marks)

(b) Show that

$$V = \begin{pmatrix} 0 \\ 1 \\ -1+i \end{pmatrix}$$

is an eigenvector of the matrix A found in (a). State the corresponding eigenvalue.

(35 marks)

(c) Find the three linearly independent real solutions.

(25 marks)

(d) Hence, solve the initial value problem

$$\frac{dX}{dt} = AX, \quad \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

where the matrix A is as in (a).

(25 marks)

5. A function $f(t)$ is defined by

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 2t - 3, & t \ge 4 \end{cases}$$

(a) Graph the function $f(t)$ from $t = 0$ to $t = 7$.

(10 marks)

(b) Write the function $f(t)$ in term of Heaviside function $U_a(t)$ where $U_a(t)$ is defined to be

$$U_a(t) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a. \end{cases}$$

(10 marks)

(c) Find the Laplace transform of $f(t)$.

(30 marks)

(d) Using the method of Laplace Transform, find the solution of the initial value problem

$$y'' + y' = f(t)$$

where $f(t)$ is given above.

[You may use the following partial fractions

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\frac{1}{s^3(s+1)} = \frac{1}{s^3} - \frac{1}{s^s} + \frac{1}{s} - \frac{1}{s+1}$$

and the results given in Table 1].

(50 marks)

1. (a) Nyatakan teorem kewujudan dan keunikan untuk masalah nilai awal

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Pertimbangkan masalah nilai awal

$$(2 - t^2)\frac{dy}{dt} + y = \ln(1+t), \quad y(0) = -1$$

Tentukan selang terbesar *I* supaya penyelesaian kepada masalah ini pasti wujud di dalamnya.

(40 markah)

(b) Kenalpasti *jenis* bagi persamaan pembezaan berikut, dengan itu selesaikannya.

$$\ln x \frac{dy}{dx} + \frac{1}{x} y = \frac{2}{x} \ln x .$$

(30 markah)

(c) Cari penyelesaian khusus bagi persamaan pembezaan

$$e^x \frac{dy}{dx} + xy^2 = 0$$

supaya $y \to \frac{1}{2}$ apabila $x \to \infty$.

(30 markah)

2. (a) Tunjukkan bahawa

$$(2y+2x)dy+\left(y^2+2xy+2y\right)dx=0$$

adalah BUKAN persamaan pembezaan tepat.

Dengan mendarab persamaan tersebut dengan e^x, tunjukkan bahawa persamaan tersebut menjadi tepat. Dengan yang demikian cari penyelesaian tak tersirat bagi persamaan berkenaan.

(40 markah)

(b) Pergerakan suatu sistem jisim spring dikawal oleh persamaan pembezaan homogen linear berperingkat kedua

$$y''+\gamma y'+4y=0.$$

Tentukan nilai-nilai bagi γ untuk sistem bagi penyerap kritikal. Untuk $\gamma=5$, selesaikan persamaan dengan syarat awal

$$y(0)=1,\ y'(0)=2.$$

(60 markah)

3. (a) Tuliskan bentuk yang sesuai bagi penyelesaian khusus untuk

$$y''-3y'-4y=3e^{2t}-\sin 4t$$

berasaskan kaedah koefisien tak tentu. **Tidak** perlu selesaikan untuk koefisien.

(30 markah)

(b) Selesaikan persamaan pembezaan Euler

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

Cari penyelesaian am bagi persamaan pembezaan peringkat kedua

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x\ln x$$

dengan menggunakan kaedah variasi parameter. Nyatakan selang di mana penyelesaian am tertakrif.

(70 markah)

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4. Diberi sistem persamaan pembezaan linear yang homogen

$$\frac{dx}{dt} - x = 0$$
$$\frac{dy}{dt} - y - z = 0$$
$$\frac{dz}{dt} + 2y + z = 0$$

(a) Tuliskan sistem persamaan tersebut dalam bentuk

$$\frac{dX}{dt} = AX,$$

di mana $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ dan kenalpasti matriks *A*.

(15 markah)

(b) Tunjukkan bahawa

$$V = \begin{pmatrix} 0 \\ 1 \\ -1+i \end{pmatrix}$$

adalah vektor eigen bagi matriks A dalam (a). Nyatakan nilai eigen yang bersepadan.

(35 markah)

(c) Cari tiga penyelesaian nyata yang tak bersandar secara linear.

(25 markah)

(d) Dengan yang demikian, selesaikan masalah nilai awal

$$\frac{dX}{dt} = AX, \quad \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

di mana matriks A adalah seperti dalam (a).

(25 markah)

5. Fungsi $f(t)$ ditakrifkan oleh

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 2t - 3, & t \ge 4 \end{cases}$$

(a) Grafkan fungsi $f(t)$ dari $t = 0$ ke $t = 7$.

(10 markah)

(b) Tuliskan fungsi $f(t)$ dalam sebutan fungsi Heaviside $U_a(t)$ di mana $U_a(t)$ adalah ditakrifkan oleh

$$U_a(t) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a. \end{cases}$$

(10 markah)

(c) Cari jelmaan Laplace bagi $f(t)$.

(30 markah)

(d) Dengan menggunakan kaedah jelmaan Laplace, cari penyelesaian bagi masalah nilai awal

$$y'' + y' = f(t)$$

di mana $f(t)$ diberi seperti di atas.

[Anda boleh menggunakan pecahan separa berikut

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\frac{1}{s^3(s+1)} = \frac{1}{s^3} - \frac{1}{s^s} + \frac{1}{s} - \frac{1}{s+1}$$

dan keputusan yang diberi dalam Jadual 1].

(50 markah)

Table 1/Jadual 1

Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. t^n, $n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin\ at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos\ at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh\ at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh\ at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin\ bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos\ bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct} f(t)$	$f(s-c)$
15. $f'(t)$	$sF(s) - f(0)$
16. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$

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